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# Application of Fractional Laplace Transform Method in Solving Linear Systems of Fractional Differential Equations

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*Abstract:* **Based on Jumarie type of Riemann-Liouville (R-L) fractional calculus, this paper provides several examples to illustrate how to use fractional Laplace transform to find the solution of linear system of fractional differential equations. A new multiplication of fractional analytic functions plays an important role in this article. In fact, our results are generalizations of those results in ordinary differential equations.**

*Keywords:* **Jumarie type of R-L fractional calculus, fractional Laplace transform, linear system of fractional differential equations, new multiplication, fractional analytic functions.**

## I. INTRODUCTION

In 1695, the concept of fractional derivative first appeared in a famous letter between L'Hospital and Leibniz. Many great mathematicians have further developed this field, such as Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Riemann, Hardy, Littlewood, and Weyl. In the last decades, fractional calculus has played a very important role in physics, dynamics, electrical engineering, viscoelasticity, economics, biology, control theory, and other fields [1-10]. However, the definition of fractional derivative is not unique. Common definitions include Riemann Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [11-15]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with traditional calculus.

In this paper, based on Jumarie's modification of R-L fractional calculus, some examples are provided to illustrate how to use fractional Laplace transform method to solve linear system of fractional differential equations. A new multiplication of fractional analytic functions plays an important role in this paper. Moreover, our results are natural generalizations of those results in ordinary differential equations.

## II. DEFINITIONS AND PROPERTIES

Firstly, we introduce the fractional calculus used in this paper and some important properties.

**Definition 2.1** ([16])**:** Suppose that $0 < \alpha \leq 1$, and $t_0$ is a real number. The Jumarie's modified R-L $\alpha$-fractional derivative is defined by

$$\left({}_{t_0}D_t^{\alpha}\right)[f(t)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{t_0}^{t}\frac{f(x)-f(t_0)}{(t-x)^{\alpha}}dx, \tag{1}$$

And the Jumarie type of R-L $\alpha$-fractional integral is defined by

$$\left({}_{t_0}I_t^{\alpha}\right)[f(t)] = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{f(x)}{(t-x)^{1-\alpha}}dx\ , \tag{2}$$

where $\Gamma(\ )$ is the gamma function.

**Proposition 2.2** ([17])**:** *If* $\alpha, \beta, t_0, c$ *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{t_0}D_t^{\alpha}\right)\left[(t-t_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-t_0)^{\beta-\alpha}, \tag{3}$$

*and*

$$\left({}_{t_0}D_t^{\alpha}\right)[c] = 0. \tag{4}$$

In the following, we introduce the definition of fractional analytic function.

**Definition 2.3** ([18])**:** Assume that $t, t_0$, and $a_k$ are real numbers for all $k$, $t_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an $\alpha$-fractional power series, that is, $f_\alpha(t^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}$ on some open interval containing $t_0$, then we say that $f_\alpha(t^\alpha)$ is $\alpha$-fractional analytic at $x_0$. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is $\alpha$-fractional analytic at every point in open interval $(a,b)$, then $f_\alpha$ is called an $\alpha$-fractional analytic function on $[a,b]$.

**Definition 2.4** ([19])**:** If $0 < \alpha \leq 1$, $t_0$ is a real number, and $f_\alpha(t^\alpha)$ and $g_\alpha(t^\alpha)$ are two $\alpha$-fractional analytic functions defined on an interval containing $t_0$ ,

$$f_\alpha(t^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}, \tag{5}$$

$$g_\alpha(t^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}. \tag{6}$$

Then

$$\begin{aligned} & f_\alpha(t^\alpha) \otimes g_\alpha(t^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\right)(t-t_0)^{k\alpha}. \end{aligned} \tag{7}$$

In other words,

$$\begin{aligned} & f_\alpha(t^\alpha) \otimes g_\alpha(t^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k} \otimes \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}. \end{aligned} \tag{8}$$

**Definition 2.5 :** The Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \tag{9}$$

where $\alpha$ is a real number, $\alpha \geq 0$, and $z$ is a complex number.

**Definition 2.6** ([19])**:** If $0 < \alpha \leq 1$, and $p, t$ are real numbers. $E_\alpha(pt^\alpha) = \sum_{k=0}^{\infty} \frac{p^k t^{k\alpha}}{\Gamma(k\alpha+1)}$ is called $\alpha$-fractional exponential function, and the $\alpha$-fractional cosine and sine function are defined as follows:

$$cos_\alpha(pt^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k p^{2k} t^{2k\alpha}}{\Gamma(2k\alpha+1)}, \tag{10}$$

and

$$sin_\alpha(pt^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k p^{2k+1} t^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}. \tag{11}$$

The following is the definition of fractional Laplace transform.

**Definition 2.7** ([20])**:** Suppose that $0 < \alpha \leq 1$, $s$ is a real variable, and $f_\alpha(t^\alpha)$ is an $\alpha$-fractional analytic functions defined for all $t \geq 0$. The function $F_\alpha(s)$ defined by the $\alpha$-fractional improper integral $\left({}_0I^\alpha_{+\infty}\right)[E_\alpha(-st^\alpha)\otimes f_\alpha(t^\alpha)]$ is called the $\alpha$-fractional Laplace transform of the function $f_\alpha$, and is denoted by $L_\alpha\{f_\alpha(t^\alpha)\}$. That is,

$$F_\alpha(s) = L_\alpha\{f_\alpha(t^\alpha)\} = \left({}_0I^\alpha_{+\infty}\right)[E_\alpha(-st^\alpha)\otimes f_\alpha(t^\alpha)]. \tag{12}$$

In the following, we introduce the major results in this paper.

**Proposition 2.8** (linearity of fractional Laplace transform)**:** *The fractional Laplace transform is a linear operation, that is, for any fractional analytic functions $f_\alpha(t^\alpha)$ and $g_\alpha(t^\alpha)$ whose fractional Laplace transforms exist, then for any constants $a$ and $b$, the fractional Laplace transform $af_\alpha(t^\alpha) + b\,g_\alpha(t^\alpha)$ exists and*

$$L_\alpha\{a\,f_\alpha(t^\alpha) + b\,g_\alpha(t^\alpha)\} = aL_\alpha\{f_\alpha(t^\alpha)\} + bL_\alpha\{g_\alpha(t^\alpha)\}. \tag{13}$$

**Theorem 2.9** ([20])**:** *If $0 < \alpha \leq 1$, $s, t, p, \omega$ are real numbers, $t \geq 0$, and $n$ is a positive integer. Then*

$$L_\alpha\{1\} = \frac{1}{s}, \text{ where } s > 0 \tag{14}$$

$$L_\alpha\{E_\alpha(pt^\alpha)\} = \frac{1}{s-p}, \text{ where } s > p \tag{15}$$

$$L_\alpha\{t^{n\alpha}\} = \frac{\Gamma(n\alpha+1)}{s^{n+1}}, \text{ where } s > 0 \tag{16}$$

$$L_\alpha\{cos_\alpha(\omega t^\alpha)\} = \frac{s}{s^2+\omega^2}, \text{ where } s > 0 \tag{17}$$

$$L_\alpha\{sin_\alpha(\omega t^\alpha)\} = \frac{\omega}{s^2+\omega^2}, \text{ where } s > 0 \tag{18}$$

$$L_\alpha\{cosh_\alpha(pt^\alpha)\} = \frac{s}{s^2-p^2}, \text{ where } s > |p| \tag{19}$$

$$L_\alpha\{sinh_\alpha(pt^\alpha)\} = \frac{p}{s^2-p^2}, \text{ where } s > |p| \tag{20}$$

$$L_\alpha\{E_\alpha(pt^\alpha)\otimes cos_\alpha(\omega t^\alpha)\} = \frac{s-p}{(s-p)^2+\omega^2}, \text{ where } s > p \tag{21}$$

$$L_\alpha\{E_\alpha(pt^\alpha)\otimes sin_\alpha(\omega t^\alpha)\} = \frac{\omega}{(s-p)^2+\omega^2}, \text{ where } s > p \tag{22}$$

## III. RESULTS AND EXAMPLES

In this section, the main results are introduced and we provide some examples to illustrate how to use fractional Laplace transform to solve linear system of fractional differential equations.

**Definition 3.1:** If $0 < \alpha \leq 1$, $n$ is a positive integer. The matrix form of linear system of fractional differential equations is

$$\left({}_0D^\alpha_t\right)[x_\alpha(t^\alpha)] = A[x_\alpha(t^\alpha)]. \tag{23}$$

Where $x_\alpha(t^\alpha) = \begin{bmatrix} x^1_\alpha(t^\alpha) \\ x^2_\alpha(t^\alpha) \\ \vdots \\ x^n_\alpha(t^\alpha) \end{bmatrix}$ and $A$ is an $n \times n$ constant matrix.

**Theorem 3.2:** *Let $0 < \alpha \leq 1$ and $I$ be the $n \times n$ identity matrix. Then the initial-value problem of linear system of fractional differential equations*

$$\left({}_0D^\alpha_t\right)[x_\alpha(t^\alpha)] = A[x_\alpha(t^\alpha)] \text{ with initial value } x_\alpha(0),$$

*has the solution*

$$x_\alpha(t^\alpha) = L^{-1}_\alpha\{(sI - A)^{-1}\}x_\alpha(0). \tag{24}$$

*Where $L^{-1}_\alpha$ is the inverse $\alpha$-fractional Laplace transform.*

**Proof** By $\alpha$-fractional Laplace transform, we obtain

$$sX_\alpha(s) - x_\alpha(0) = AX_\alpha(s). \tag{25}$$

It follows that

$$(sI - A)X_\alpha(s) = x_\alpha(0). \tag{26}$$

And hence,

$$X_\alpha(s) = (sI - A)^{-1}x_\alpha(0). \tag{27}$$

Finally, we have

$$x_\alpha(t^\alpha) = L_\alpha^{-1}\{X_\alpha(s)\} = L_\alpha^{-1}\{(sI - A)^{-1}x_\alpha(0)\} = L_\alpha^{-1}\{(sI - A)^{-1}\}x_\alpha(0). \qquad \text{Q.e.d.}$$

**Example 3.3:** If $0 < \alpha \leq 1$.Solve the initial-value problem of linear system of $\alpha$-fractional differential equations

$$\left({}_0D_t^\alpha\right)[x_\alpha^1(t^\alpha)] = -3 \cdot x_\alpha^1(t^\alpha) - x_\alpha^2(t^\alpha), \quad x_\alpha^1(0) = 1, \tag{28}$$

$$\left({}_0D_t^\alpha\right)[x_\alpha^2(t^\alpha)] = 2 \cdot x_\alpha^1(t^\alpha), \quad x_\alpha^2(0) = 0. \tag{29}$$

**Solution** Since the coefficient matrix of this linear system is

$$A = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}. \tag{30}$$

It follows that

$$\begin{aligned} & sI - A \\ &= s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}. \end{aligned} \tag{31}$$

And hence,

$$\begin{aligned} & (sI - A)^{-1} \\ &= \frac{1}{s^2+3s+2}\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \frac{-1}{(s+1)(s+2)} \\ \frac{2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix} \\ &= \begin{bmatrix} \frac{-1}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{1}{s+2} \\ \frac{2}{s+1} + \frac{-2}{s+2} & \frac{2}{s+1} + \frac{-1}{s+2} \end{bmatrix}. \end{aligned} \tag{32}$$

Therefore, we obtain

$$\begin{aligned} & L_\alpha^{-1}\{(sI - A)^{-1}\} \\ &= L_\alpha^{-1}\left\{\begin{bmatrix} \frac{-1}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{1}{s+2} \\ \frac{2}{s+1} + \frac{-2}{s+2} & \frac{2}{s+1} + \frac{-1}{s+2} \end{bmatrix}\right\} \\ &= \begin{bmatrix} -E_\alpha(-t^\alpha) + 2 \cdot E_\alpha(-2t^\alpha) & -E_\alpha(-t^\alpha) + E_\alpha(-2t^\alpha) \\ 2 \cdot E_\alpha(-t^\alpha) - 2 \cdot E_\alpha(-2t^\alpha) & 2 \cdot E_\alpha(-t^\alpha) - E_\alpha(-2t^\alpha) \end{bmatrix}. \end{aligned} \tag{33}$$

So, the solution of this linear system is

$$\begin{aligned} x_\alpha(t^\alpha) &= L_\alpha^{-1}\{(sI - A)^{-1}\}x_\alpha(0) \\ &= \begin{bmatrix} -E_\alpha(-t^\alpha) + 2 \cdot E_\alpha(-2t^\alpha) & -E_\alpha(-t^\alpha) + E_\alpha(-2t^\alpha) \\ 2 \cdot E_\alpha(-t^\alpha) - 2 \cdot E_\alpha(-2t^\alpha) & 2 \cdot E_\alpha(-t^\alpha) - E_\alpha(-2t^\alpha) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} -E_\alpha(-t^\alpha) + 2 \cdot E_\alpha(-2t^\alpha) \\ 2 \cdot E_\alpha(-t^\alpha) - 2 \cdot E_\alpha(-2t^\alpha) \end{bmatrix}. \tag{34}$$

That is,

$$x_\alpha^1(t^\alpha) = -E_\alpha(-t^\alpha) + 2 \cdot E_\alpha(-2t^\alpha), \tag{35}$$

$$x_\alpha^2(t^\alpha) = 2 \cdot E_\alpha(-t^\alpha) - 2 \cdot E_\alpha(-2t^\alpha). \tag{36}$$

**Example 3.4:** Let $0 < \alpha \leq 1$.Find the solution of the initial-value problem of linear system of $\alpha$-fractional differential equations

$$\left({}_0D_t^\alpha\right) \begin{bmatrix} x_\alpha^1(t^\alpha) \\ x_\alpha^2(t^\alpha) \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_\alpha^1(t^\alpha) \\ x_\alpha^2(t^\alpha) \end{bmatrix}, \quad \begin{bmatrix} x_\alpha^1(0) \\ x_\alpha^2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{37}$$

**Solution** Since

$$\begin{aligned} & sI - A \\ &= s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} s-3 & 4 \\ -1 & s+1 \end{bmatrix}. \end{aligned} \tag{38}$$

It follows that

$$\begin{aligned} & (sI - A)^{-1} \\ &= \frac{1}{s^2-2s+1}\begin{bmatrix} s+1 & -4 \\ 1 & s-3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{s+1}{(s-1)^2} & \frac{-4}{(s-1)^2} \\ \frac{1}{(s-1)^2} & \frac{s-3}{(s-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{s-1} + \frac{2}{(s-1)^2} & \frac{-4}{(s-1)^2} \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} + \frac{-2}{(s-1)^2} \end{bmatrix}. \end{aligned} \tag{39}$$

Thus,

$$\begin{aligned} & L_\alpha^{-1}\{(sI - A)^{-1}\} \\ &= L_\alpha^{-1}\left\{\begin{bmatrix} \frac{1}{s-1} + \frac{2}{(s-1)^2} & \frac{-4}{(s-1)^2} \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} + \frac{-2}{(s-1)^2} \end{bmatrix}\right\} \\ &= \begin{bmatrix} E_\alpha(t^\alpha) + 2 \cdot \frac{1}{\Gamma(\alpha+1)} t^\alpha \otimes E_\alpha(t^\alpha) & -4 \cdot \frac{1}{\Gamma(\alpha+1)} t^\alpha \otimes E_\alpha(t^\alpha) \\ \frac{1}{\Gamma(\alpha+1)} t^\alpha \otimes E_\alpha(t^\alpha) & E_\alpha(t^\alpha) - 2 \cdot \frac{1}{\Gamma(\alpha+1)} t^\alpha \otimes E_\alpha(t^\alpha) \end{bmatrix}. \end{aligned} \tag{40}$$

Therefore, the solution of this linear system is

$$\begin{aligned} x_\alpha(t^\alpha) &= L_\alpha^{-1}\{(sI - A)^{-1}\} x_\alpha(0) \\ &= \begin{bmatrix} E_\alpha(t^\alpha) + 2 \cdot \frac{1}{\Gamma(\alpha + 1)} t^\alpha \otimes E_\alpha(t^\alpha) & -4 \cdot \frac{1}{\Gamma(\alpha + 1)} t^\alpha \otimes E_\alpha(t^\alpha) \\ \frac{1}{\Gamma(\alpha + 1)} t^\alpha \otimes E_\alpha(t^\alpha) & E_\alpha(t^\alpha) - 2 \cdot \frac{1}{\Gamma(\alpha + 1)} t^\alpha \otimes E_\alpha(t^\alpha) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} E_\alpha(t^\alpha) - 2 \cdot \frac{1}{\Gamma(\alpha+1)} t^\alpha \otimes E_\alpha(t^\alpha) \\ E_\alpha(t^\alpha) - \frac{1}{\Gamma(\alpha+1)} t^\alpha \otimes E_\alpha(t^\alpha) \end{bmatrix}. \end{aligned} \tag{41}$$

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Hence,

$$x_\alpha^1(t^\alpha) = \left(1 - 2 \cdot \frac{1}{\Gamma(\alpha+1)} t^\alpha\right) \otimes E_\alpha(t^\alpha), \tag{42}$$

$$x_\alpha^2(t^\alpha) = \left(1 - \frac{1}{\Gamma(\alpha+1)} t^\alpha\right) \otimes E_\alpha(t^\alpha). \tag{43}$$

**Example 3.5:** Suppose that $0 < \alpha \leq 1$.Find the solution of the initial-value problem of linear system of $\alpha$-fractional differential equations

$$\left({}_0D_t^\alpha\right)\begin{bmatrix} x_\alpha^1(t^\alpha) \\ x_\alpha^2(t^\alpha) \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} x_\alpha^1(t^\alpha) \\ x_\alpha^2(t^\alpha) \end{bmatrix}, \quad \begin{bmatrix} x_\alpha^1(0) \\ x_\alpha^2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{44}$$

**Solution** Since

$$\begin{aligned} & sI - A \\ & = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ -3 & -1 \end{bmatrix} \\ & = \begin{bmatrix} s-5 & -6 \\ 3 & s+1 \end{bmatrix}. \end{aligned} \tag{45}$$

It follows that

$$\begin{aligned} & (sI - A)^{-1} \\ & = \frac{1}{s^2 - 4s + 13}\begin{bmatrix} s+1 & 6 \\ -3 & s-5 \end{bmatrix} \\ & = \begin{bmatrix} \frac{s+1}{(s-2)^2+3^2} & \frac{6}{(s-2)^2+3^2} \\ \frac{-3}{(s-2)^2+3^2} & \frac{s-5}{(s-2)^2+3^2} \end{bmatrix} \\ & = \begin{bmatrix} \frac{s-2}{(s-2)^2+3^2} + \frac{3}{(s-2)^2+3^2} & 2 \cdot \frac{3}{(s-2)^2+3^2} \\ -\frac{3}{(s-2)^2+3^2} & \frac{s-2}{(s-2)^2+3^2} - \frac{3}{(s-2)^2+3^2} \end{bmatrix}. \end{aligned} \tag{46}$$

Hence,

$$\begin{aligned} & L_\alpha^{-1}\{(sI - A)^{-1}\} \\ & = L_\alpha^{-1}\left\{\begin{bmatrix} \frac{s-2}{(s-2)^2+3^2} + \frac{3}{(s-2)^2+3^2} & 2 \cdot \frac{3}{(s-2)^2+3^2} \\ -\frac{3}{(s-2)^2+3^2} & \frac{s-2}{(s-2)^2+3^2} - \frac{3}{(s-2)^2+3^2} \end{bmatrix}\right\} \\ & = \begin{bmatrix} E_\alpha(2t^\alpha) \otimes cos_\alpha(3t^\alpha) + E_\alpha(2t^\alpha) \otimes sin_\alpha(3t^\alpha) & 2 \cdot E_\alpha(2t^\alpha) \otimes sin_\alpha(3t^\alpha) \\ -E_\alpha(2t^\alpha) \otimes sin_\alpha(3t^\alpha) & E_\alpha(2t^\alpha) \otimes cos_\alpha(3t^\alpha) - E_\alpha(2t^\alpha) \otimes sin_\alpha(3t^\alpha) \end{bmatrix}. \end{aligned} \tag{47}$$

Thus, the solution of this linear system is

$$\begin{aligned} x_\alpha(t^\alpha) &= L_\alpha^{-1}\{(sI - A)^{-1}\} x_\alpha(0) \\ &= \begin{bmatrix} E_\alpha(2t^\alpha) \otimes cos_\alpha(3t^\alpha) + E_\alpha(2t^\alpha) \otimes sin_\alpha(3t^\alpha) & 2 \cdot E_\alpha(2t^\alpha) \otimes sin_\alpha(3t^\alpha) \\ -E_\alpha(2t^\alpha) \otimes sin_\alpha(3t^\alpha) & E_\alpha(2t^\alpha) \otimes cos_\alpha(3t^\alpha) - E_\alpha(2t^\alpha) \otimes sin_\alpha(3t^\alpha) \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -E_\alpha(2t^\alpha) \otimes cos_\alpha(3t^\alpha) + E_\alpha(2t^\alpha) \otimes sin_\alpha(3t^\alpha) \\ E_\alpha(2t^\alpha) \otimes cos_\alpha(3t^\alpha) \end{bmatrix}. \end{aligned} \tag{48}$$

Finally, we obtain

$$x_\alpha^1(t^\alpha) = \left(-cos_\alpha(3t^\alpha) + sin_\alpha(3t^\alpha)\right) \otimes E_\alpha(2t^\alpha), \tag{49}$$

$$x_\alpha^2(t^\alpha) = cos_\alpha(3t^\alpha) \otimes E_\alpha(t^\alpha). \tag{50}$$

## IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional calculus, some examples are given to illustrate how to use fractional Laplace transform to solve linear system of fractional differential equations. A new multiplication of fractional analytic functions plays an important role in this paper. In fact, the results we obtained are generalizations of those results in ordinary differential equations. In the future, we will continue to use fractional Laplace transform to study the problems in fractional differential equations and engineering mathematics.

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